Time-dependent Schrödinger equation in dimension $k+1$: explicit and rational solutions via GBDT and multinodes

A.L. Sakhnovich

E-mail address: Oleksandr.Sakhnovych@univie.ac.at

Abstract

A version of the binary Darboux transformation is constructed for non-stationary Schrödinger equation in dimension $k+1$, where k is the number of space variables, $k \geq 1$. This is an iterated GBDT version. New families of non-singular and rational potentials and solutions are obtained. Some results are new for the case that $k = 1$ too. A certain generalization of a colligation introduced by M.S. Livšic and of the S-node introduced by L.A. Sakhnovich is successfully used in our construction.

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1 Introduction

The time-dependent, or non-stationary, Schrödinger (TDS) equation is one of the most well-known and important equations in physics. We shall consider the vector case of the Schrödinger equation and an arbitrary coefficient α

($\alpha \in \mathbb{C},\ \alpha \neq 0$):

$$Hu = 0, \quad H := \alpha\frac{\partial}{\partial t} + \Delta - q(x,t), \quad u \in \mathbb{C}^p, \tag{1.1}$$

where Δ denotes the Laplacian with respect to the spatial variables $x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, q is a $p \times p$ matrix function, $\mathbb{C}$ stands for the complex plane, and $\mathbb{R}$ stands for the real axis. Explicit solutions of (1.1) are of essential and permanent interest and various methods were applied to construct them. For example, rational slowly decaying soliton solutions (lumps) of the Kadomtsev-Petviashvili (KP) and time-dependent Schrödinger (TDS) equations were first found in [28]. Since then they were actively studied as well as rational solutions of other integrable equations (see, e.g., [1,13,14,22,41,48,49,51] and references therein).

Following seminal works of Bäcklund, Darboux, and Jacobi, different kinds of Darboux transformations and related commutation and factorization methods are fruitfully used to obtain explicit solutions of linear and nonlinear equations (see, e.g., [9–11,16–21,29,31,32,40,50] and numerous references therein). In particular, matrix and operator identities were widely used in these constructions (see [6,8,12,15,27,29,35,37–39,41,45] for various results, discussions and references).

Darboux transformations proved to be especially useful for the construction of explicit solutions of the TDS equation in dimension $1+1$ (and, correspondingly, for the construction of explicit solutions of the KP equation with two spatial variables). A singular (non-binary) Darboux transformation was used for that purpose in [30] and a binary Darboux transformation for the scalar TDS appeared in the well-known book [31, Section 2.4]. Further important results on the TDS equation in dimension $1+1$ can be found in [1,3,7,48] (see also [47] and references therein for generalized TDS equations).

The first discussion on the Darboux transformation for TDS with $k > 1$ spatial variables that we could find, was in the work [33] (Sabatier, 1991). In spite of interesting publications (e.g, [2,34]), the case of linear equations with k spatial variables ($k > 1$) is much more difficult (and much less is done, even for the singular Darboux transformation), especially so for $k > 2$ (see [34,46] for some explanations).

In this paper we construct explicit and rational solutions of the TDS equation for $k > 1$ and $k > 2$ as well. We apply the generalized Bäcklund-Darboux approach (GBDT) from [15,27,35,37–39,41] (see further references and some comparative discussions on this method in [9,20,40]). Corresponding results for the case $k = 1$ were announced in [38] and the present paper contains proofs, which are valid for $k = 1$ too. GBDT is partially based on the operator identity (also called S-node) method [42–44], which, in its turn, takes its roots in the characteristic matrix function and operator colligations introduced by M.S. Livšic [23] (see also [24]). In his later works M.S. Livšic studied a greatly more complicated case of colligations with several (instead of one) commuting operators [25] (see also [4,5,26] and references therein). Correspondingly, for the case of k spatial variables we need an S-node with k matrix identities, which we call S-*multinode*.

In Section 2 we describe GBDT for the TDS equation. S-multinodes are introduced in Section 3. We use GBDT and multinodes in Section 3 to construct explicitly solutions and potentials of the TDS equation and consider examples. Conditions for non-singular and rational solutions and potentials and concrete examples are given in Section 4.

We use $\mathbb{N}$ to denote the set of natural numbers, σ to denote spectrum, and $\overline{\alpha}$ to denote the complex conjugate of α. The notatation $\mathrm{Rank}(A)$ stands for the rank of a matrix A, A^* is the matrix adjoint to A, and col denotes a column.

2 GBDT for the TDS equation

Let $H = \alpha\frac{\partial}{\partial t} + \Delta - q$ be some TDS equation, which we call *initial*, and let $\Psi(x,t)$ and $\Phi(x,t)$ be block rows of $n \times p$ blocks Ψ_r and Φ_r, respectively. Here $n \in \mathbb{N}$ is fixed and $0 \leq r \leq k$. It is required that Ψ satisfies equations

$$H\Psi_0^* = 0, \quad \Psi_r^* = \frac{\partial}{\partial x_r}\Psi_0^* \quad (1 \leq r \leq k), \tag{2.1}$$

where H is applied to Ψ_0^* columnwise. In other words, Ψ^* satisfies a first order differential system, which is equivalent to TDS:

$$L\Psi^* = 0, \quad L := \alpha\theta_0 \frac{\partial}{\partial t} + \sum_{r=1}^{k} \theta_r \frac{\partial}{\partial x_r} - \theta_{k+1}; \tag{2.2}$$

$$\theta_0 := e_k e_0^*, \qquad \theta_r := e_{r-1}e_0^* + e_k e_r^* \quad (1 \le r \le k),$$
$$\theta_{k+1}(x,t) := e_k q(x,t) e_0^* + \{\delta_{i,j-1} I_p\}_{i,j=0}^k, \tag{2.3}$$

where $\delta_{i,j}$ is the Kronecker's delta, I_p is the $p \times p$ identity matrix, and $(k+1)p \times p$ matrices e_i are given by the equilities $e_i := \{\delta_{i,j} I_p\}_{j=0}^k$. We require

$$\Phi_r = \frac{\partial}{\partial x_r}\Phi_0 \qquad (1 \le r \le k), \tag{2.4}$$

and Φ_0 shall be discussed a bit later.

An $n \times n$ matrix function $\mathcal{S}$, which we define via Ψ and Φ:

$$\frac{\partial}{\partial x_r}\mathcal{S}(x,t) = \Phi_0(x,t)\nu_r\Psi_0(x,t)^* \qquad (1 \le r \le k), \tag{2.5}$$

$$\frac{\partial}{\partial t}\mathcal{S}(x,t) = \alpha^{-1}\sum_{r=1}^{k}\big(\Phi_r(x,t)\nu_r\Psi_0(x,t)^* - \Phi_0(x,t)\nu_r\Psi_r(x,t)^*\big), \tag{2.6}$$

is very important in GBDT. Here ν_r are some $p \times p$ matrices. For linear equations depending on one variable and nonlinear equations depending on two variables, the analog of $\mathcal{S}$ is denoted by S and the so called Darboux matrix is presented as the transfer matrix function of the corresponding S-node. The equality $\widetilde{\Psi}_0^* = \Psi_0^*\mathcal{S}^{-1}$ for the solution $\widetilde{\Psi}_0^*$ of the transformed TDS $\widetilde{H}f = 0$ holds also in our case (see Theorem 2.1 below).

Since GBDT is a kind of binary Darboux transform, the matrix function Φ_0 should satisfy some dual to TDS differential equation $H_d\Phi_0 = 0$. In view of (2.1), (2.4), (2.5), and (2.6), for the case that

$$H_d f = \alpha\frac{\partial}{\partial t}f(x,t) - \Delta f(x,t) - f(x,t)q_d(x,t), \qquad q_d\nu_i = \nu_i q \quad (1 \le i \le k), \tag{2.7}$$

where f is a row vector function and q_d is a $p \times p$ matrix function, we have

$$\mathcal{S}_{tx_i} = \alpha^{-1} \sum_{r=1}^{k} \Big(\Phi_r \nu_r \Psi_i^* - \Phi_i \nu_r \Psi_r^* + \Big(\frac{\partial}{\partial x_i} \Phi_r \Big) \nu_r \Psi_0^* - \Phi_0 \nu_r \Big(\frac{\partial}{\partial x_i} \Psi_r^* \Big) \Big), \tag{2.8}$$

$$\begin{aligned} \mathcal{S}_{x_i t} &= \alpha^{-1} \big((\Delta \Phi_0) \nu_i \Psi_0^* - \Phi_0 \nu_i (\Delta \Psi_0^*) + \Phi_0 (\nu_i q - q_d \nu_i) \Psi_0^* \big) \\ &= \alpha^{-1} \big((\Delta \Phi_0) \nu_i \Psi_0^* - \Phi_0 \nu_i (\Delta \Psi_0^*) \big), \qquad \mathcal{S}_{x_i t} := \frac{\partial}{\partial t} \frac{\partial}{\partial x_i} \mathcal{S}. \end{aligned} \tag{2.9}$$

Because of (2.8) and (2.9), the compatibility condition $\mathcal{S}_{x_1 t} = \mathcal{S}_{t x_1}$ for equations (2.5) and (2.6) is fulfilled for the case that $k = 1$. However, for $k > 1$ the situation is more complicated, and we just assume the existance of $\mathcal{S}$, satisfying (2.5) and (2.6), and don't assume (2.7) in our theorem below.

Theorem 2.1 *Let matrix functions Ψ, Φ, and $\mathcal{S}$ satisfy relations* (2.1), (2.4), *and* (2.5), (2.6), *respectively. Then, in the points of invertibility of $\mathcal{S}$, the matrix function*

$$\widetilde{\Psi}_0^* := \Psi_0^* \mathcal{S}^{-1} \tag{2.10}$$

satisfies the transformed TDS equation:

$$\widetilde{H} \widetilde{\Psi}_0^* = 0, \quad \widetilde{H} := \alpha \frac{\partial}{\partial t} + \Delta - \widetilde{q}(x,t), \tag{2.11}$$

where

$$\widetilde{q}(x,t) := q(x,t) - 2 \sum_{r=1}^{k} \frac{\partial}{\partial x_r} \big(\Psi_0(x,t)^* \mathcal{S}(x,t)^{-1} \Phi_0(x,t) \big) \nu_r. \tag{2.12}$$

P r o o f. Taking into account (2.1) and definitions of H, Ψ_0, and $\widetilde{H}$ in (1.1), (2.10), and (2.11), respectively, we get

$$\widetilde{H} \widetilde{\Psi}_0^* = (q - \widetilde{q}) \widetilde{\Psi}_0^* - \alpha \Psi_0^* \mathcal{S}^{-1} \mathcal{S}_t \mathcal{S}^{-1} + \Psi_0^* \Delta(\mathcal{S}^{-1}) - 2 \sum_{r=1}^{k} \Psi_r^* \mathcal{S}^{-1} \mathcal{S}_{x_r} \mathcal{S}^{-1}. \tag{2.13}$$

Because of (2.5), we have

$$\Delta(\mathcal{S}^{-1}) = \sum_{r=1}^{k} \mathcal{S}^{-1}\big(2\mathcal{S}_{x_r}\mathcal{S}^{-1}\mathcal{S}_{x_r} - \Phi_r\nu_r\Psi_0^* - \Phi_0\nu_r\Psi_r^*\big)\mathcal{S}^{-1}. \tag{2.14}$$

Finally, using formulas (2.6) and (2.14) and reducing similar terms, we rewrite (2.13) as

$$\begin{aligned}\widetilde{H}\widetilde{\Psi}_0^* =&(q-\widetilde{q})\widetilde{\Psi}_0^* - \Psi_0^*\mathcal{S}^{-1}\sum_{r=1}^{k}\Phi_r\nu_r\widetilde{\Psi}_0^* + 2\Psi_0^*\mathcal{S}^{-1}\Phi_0\sum_{r=1}^{k}\nu_r\Psi_0^*\mathcal{S}^{-1}\Phi_0\nu_r\widetilde{\Psi}_0^* \\ &- \Psi_0^*\mathcal{S}^{-1}\sum_{r=1}^{k}\Phi_r\nu_r\widetilde{\Psi}_0^* - 2\sum_{r=1}^{k}\Psi_r^*\mathcal{S}^{-1}\Phi_0\nu_r\widetilde{\Psi}_0^*.\end{aligned} \tag{2.15}$$

Since $\Phi_r = (\Phi_0)_{x_r}$, $\Psi_r = (\Psi_0)_{x_r}$, and (2.5) holds, it follows from (2.15) that for $\widetilde{q}$ given by (2.12) the equality $\widetilde{H}\widetilde{\Psi}_0^* = 0$ is true. ■

3 Multinodes and explicit solutions

Definition 3.1 *By a matrix S-multinode (or, more precisely, by S_k-node $\{k, A, B, R, \nu, C_\Phi, C_\Psi\}$) we call a set of matrices, which consists of $N \times N$ commuting matrices A_r $(1 \le r \le k)$, of $N \times N$ commuting matrices B_r $(1 \le r \le k)$, of $p \times p$ matrices ν_r $(1 \le r \le k)$, and of an $N \times N$ matrix R, an $N \times p$ matrix C_Φ, and a $p \times N$ matrix C_Ψ, such that the matrix identities*

$$A_r R - R B_r = C_\Phi \nu_r C_\Psi, \qquad 1 \le r \le k \tag{3.1}$$

hold. An operator S-multinode is defined in the same way.

For the case that $k = 1$ this definition coincides with the definition of an S-node [42–44], and for the case that $R = I_N$ and $B_r = A_r^*$ our definition coincides with the definition of a colligation from [25].

In this section we treat the case $q \equiv 0$, that is,

$$\widetilde{q}(x,t) = -2\sum_{r=1}^{k}\frac{\partial}{\partial x_r}\big(\Psi_0(x,t)^*\mathcal{S}(x,t)^{-1}\Phi_0(x,t)\big)\nu_r. \tag{3.2}$$

Theorem 3.2 *Let an $n \times N$ matrix $\widehat{C}_\Phi$, an $N \times n$ matrix $\widehat{C}_\Psi$, an $n \times n$ matrix $\mathcal{S}_0$, and a matrix S_k-node $\big\{k, A, B, R, \nu, C_\Phi, C_\Psi\big\}$ be given. Then the matrix functions*

$$\Phi_0(x,t) = \widehat{C}_\Phi \mathrm{e}_A(x,t) C_\Phi, \quad \mathrm{e}_A(x,t) := \exp\Big\{\big(\sum_{r=1}^{k} x_r A_r\big) + \alpha^{-1} t \big(\sum_{r=1}^{k} A_r^2\big)\Big\}, \tag{3.3}$$

$$\Psi_0(x,t)^* = C_\Psi \mathrm{e}_B(-x,-t)\widehat{C}_\Psi, \quad \mathcal{S} = \widehat{C}_\Phi \mathrm{e}_A(x,t) R \mathrm{e}_B(-x,-t)\widehat{C}_\Psi + \mathcal{S}_0 \tag{3.4}$$

satisfy conditions of Theorem 2.1, where the initial TDS equation is chosen so that $q \equiv 0$ (i.e., the conditions on Φ and Ψ are satisfied after we standardly add $\Phi_r = (\Phi_0)_{x_r}$ and $\Psi_r = (\Psi_0)_{x_r}$).

P r o o f. It is immediate from (3.4) that $\alpha \frac{\partial}{\partial t}\Psi_0^* + \Delta \Psi_0^* = 0$, that is, $H\Psi_0^* = 0$, and so (2.1) holds. Because of (3.1), (3.3), and (3.4) we have (2.5). It remains to show that (2.6) holds. For that purpose, note that equalities (3.1) imply

$$\begin{aligned} A_r^2 R - R B_r^2 &= A_r(A_r R - R B_r) + (A_r R - R B_r) B_r \\ &= A_r C_\Phi \nu_r C_\Psi + C_\Phi \nu_r C_\Psi B_r, \quad 1 \le r \le k. \end{aligned} \tag{3.5}$$

In view of (3.4) and (3.5) we get

$$\frac{\partial}{\partial t}\mathcal{S} = \alpha^{-1}\widehat{C}_\Phi \mathrm{e}_A(x,t)\Big(\sum_{r=1}^{k} A_r C_\Phi \nu_r C_\Psi + C_\Phi \nu_r C_\Psi B_r\Big)\mathrm{e}_B(-x,-t)\widehat{C}_\Psi. \tag{3.6}$$

Since $\Phi_r = (\Phi_0)_{x_r}$ and $\Psi_r = (\Psi_0)_{x_r}$, formula (2.6) easily follows from (3.3), (3.4), and (3.6). ■

Remark 3.3 *Clearly, the equality $\alpha \frac{\partial}{\partial t}\Phi_0 - \Delta \Phi_0 = 0$ holds for Φ_0 of the form* (3.3)*, that is, we have $H_d \Phi_0 = 0$ for the case that $q_d = 0$, and so* (2.7) *holds. Hence, it goes in Theorem 3.2 about the binary GBDT.*

Recall that singular (and some stationary binary) Darboux transformations of a scalar TDS equation into the vector TDS were treated in [34]. It is easy to see that for the case that $\widetilde{q} = \{\widetilde{q}_{ij}\}_{i,j=1}^p$ and $f = \{f_i\}_{i=1}^p$ are the potential and solution, respectively, of some vector TDS equation, the functions

$$q^{sc} = \widetilde{q}_{rr} + \big(\sum_{j \ne r} \widetilde{q}_{rj} f_j\big)/f_r, \quad f^{sc} = f_r \tag{3.7}$$

are the potential and solution of a scalar TDS.

Example 3.4 *Consider the simplest case $k = 1$ (and $\nu_1 = I_p$). If A_1 and B_1 are diagonal matrices: $A_1 = \mathrm{diag}\{a_1, \dots, a_N\}$ and $B_1 = \mathrm{diag}\{b_1, \dots, b_N\}$, then identity* (3.1) *leads us to*

$$R_{ij} = (a_i - b_j)^{-1}\big(C_\Phi C_\Psi\big)_{ij}, \quad R = \{R_{ij}\}_{i,j=1}^N, \tag{3.8}$$

and relations (2.10), (3.2)-(3.4), *and* (3.8) *define solution and potential of the transformed vector TDS equation explicitly (up to inversion of matrices).*

Example 3.5 *Now, consider a simple example for the case that $1 < k \leq p$. We set*

$$A_1 = \mathrm{diag}\{a_1, \dots, a_N\}, \quad A_r = (c_r I_N - A_1)^{-1} \quad (a_i \neq c_r, \ 1 < r \leq k); \tag{3.9}$$
$$B_1 = \mathrm{diag}\{b_1, \dots, b_N\}, \quad B_r = (c_r I_N - B_1)^{-1} \quad (b_i \neq c_r, \ 1 < r \leq k); \tag{3.10}$$
$$R = \{(a_i - b_j)^{-1}\}_{i,j=1}^N \ (a_i \neq b_j); \quad \nu_r = \{\delta_{r-i}\delta_{r-j}\}_{i,j=1}^p, \ 1 \leq r \leq k; \tag{3.11}$$
$$C_\Phi = \begin{bmatrix} h & A_2 h & \dots & A_k h & \breve{C}_\Phi \end{bmatrix}, \quad h = \mathrm{col}\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}, \tag{3.12}$$
$$C_\Psi = \mathrm{col}\begin{bmatrix} h^* & h^* B_2 & \dots & h^* B_k & \breve{C}_\Psi \end{bmatrix}, \tag{3.13}$$

where col *stands for column, $\breve{C}_\Phi$ is some $N \times (p-k)$ matrix, and $\breve{C}_\Psi$ is some $(p-k) \times N$ matrix. We see that $A_1 R - R B_1 = hh^*$. Hence, the equalities $A_r R - R B_r = A_r hh^* B_r$ $(1 < r \leq k)$ hold. Therefore, relations* (3.9)-(3.13) *determine an S_k-node $\big\{k, A, B, R, \nu, C_\Phi, C_\Psi\big\}$ and the corresponding explicit expressions for solution and potential of the transformed TDS equation follow.*

GBDT with non-diagonizable matrices A_r is of interest (and has its own specifics). See, for instance, [38] where the cases that $k = 1$ and matrices A_1 are 2×2 or 3×3 Jordan cells are treated as examples. Below we present an example of a multinode, where $k > 1$ and matrices A_r are non-diagonizable.

Example 3.6 *We assume $1 \leq k \leq p$, $N > 1$ and set*

$$A_0 := \frac{\mathrm{i}}{2} I_N + \{a_{i-j}\}_{i,j=1}^N, \quad a_s := \mathrm{i} \quad \text{for } s > 0, \ a_s := 0 \quad \text{for } s \leq 0; \tag{3.14}$$
$$A_r := (c_r I_N - A_0)^{-1}, \ B_r := (c_r I_N - A_0^*)^{-1} \ (c_r \neq \pm \mathrm{i}/2, \ 1 \leq r \leq k). \tag{3.15}$$

We see that matrix A_0, and therefore matrices A_r and B_r are linear similar to Jordan cells. The matrix R is a so called cyclic Toeplitz matrix and is introduced by the equality

$$R := \{T_{i-j}\}_{i,j=1}^{N} \quad (T_s \in \mathbb{C}, \quad T_s = 0 \quad \text{for} \quad s < 0). \tag{3.16}$$

Then, the following matrix identity holds (see, e.g., [36, p. 451]):

$$A_0 R - R A_0^* = i g h^*, \quad g = \mathrm{col}\begin{bmatrix} T_0 & T_0 + T_1 & \dots & \sum_{s=0}^{N-1} T_s \end{bmatrix}, \tag{3.17}$$

where h is given in (3.12)*. Because of* (3.15) *and* (3.17)*, the identities* (3.1) *are true, where ν_r are given in* (3.11) *and*

$$C_\Phi = \mathrm{i}\begin{bmatrix} A_1 g & \dots & A_k g & \breve{C}_\Phi \end{bmatrix}, \quad C_\Psi = \mathrm{col}\begin{bmatrix} h^* B_1 & \dots & h^* B_k & \breve{C}_\Psi \end{bmatrix}. \tag{3.18}$$

That is, a multinode, where $k > 1$ and matrices A_r and B_r are non-diagonizable, is constructed.

The cases, where matrices ν_r had rank 1 and $p \times p$ matrix TDS equations with p spatial variables were included, were treated in Examples 3.5 and 3.6.

Remark 3.7 *Clearly, it is quite possible, though somewhat less convenient, to consider S_k-nodes with matrices ν_r of higher ranks in the same way. Recall also an easy transfer* (3.7) *from a matrix to a scalar TDS.*

Definition 3.1 admits an easy generalization for the case of rectangular matrices R, whereupon the proof of Theorem 3.2 does not require any changes.

Definition 3.8 *By the S_k-node $\{k, A, B, R, \nu, C_\Phi, C_\Psi\}$ (with rectangular matrix R) we call a set of matrices, which consists of $N_1 \times N_1$ commuting matrices A_r $(1 \le r \le k)$, of $N_2 \times N_2$ commuting matrices B_r $(1 \le r \le k)$, of $p \times p$ matrices ν_r $(1 \le r \le k)$, and of an $N_1 \times N_2$ matrix R, an $N_1 \times p$ matrix C_Φ, and a $p \times N_2$ matrix C_Ψ, such that the matrix identities* (3.1) *hold.*

Corollary 3.9 *Let an $n \times N_1$ matrix $\widehat{C}_\Phi$, an $N_2 \times n$ matrix $\widehat{C}_\Psi$, an $n \times n$ matrix $\mathcal{S}_0$, and a matrix S_k-node $\{k, A, B, R, \nu, C_\Phi, C_\Psi\}$ (with an $N_1 \times N_2$ matrix R) be given. Then the matrix functions Φ_0, Ψ_0, and $\mathcal{S}$, which are given by formulas* (3.3) *and* (3.4)*, satisfy equation $H_d \Phi_0 = 0$ and conditions of Theorem 2.1, where $q = q_d = 0$.*

4 Non-singular, rational, and lump potentials

In this section, we study conditions that the potentials $\widetilde{q}$ and the TDS solutions $\widetilde{\Psi}_0^*$ are non-singular and rational. Our next proposition deals with a construction of rational potentials.

Proposition 4.1 *Let an $n \times N_1$ matrix $\widehat{C}_\Phi$, an $N_2 \times n$ matrix $\widehat{C}_\Psi$, an $n \times n$ matrix $\mathcal{S}_0$, and a matrix S_k-node $\{k, A, B, R, \nu, C_\Phi, C_\Psi\}$ be given. Assume additionally that the conditions (i) below hold:*

(i) all the matrices from the set $\{A_r\} \cup \{B_r\}$ are nilpotent.

Then the solution and potential of the transformed TDS equation, which are given by formulas (2.10) and (3.2), respectively, are rational.

If, instead of (i), we assume that

$$\mathcal{S}_0 = 0; \quad A_r = \mu_r I_{N_1} + \breve{A}_r, \quad B_r = \lambda_r I_{N_2} + \breve{B}_r, \tag{4.1}$$

where $1 \leq r \leq k$; μ_r, $\lambda_r \in \mathbb{C}$, and the matrices $\breve{A}_r$ and $\breve{B}_r$ are nilpotent, then the potential $\widetilde{q}$ of the transformed TDS equation is rational.

P r o o f. In view of Corollary 3.9, the matrix functions Φ_0, Ψ_0, and $\mathcal{S}$ satisfy conditions of Theorem 2.1. Therefore, using Theorem 2.1 we see that the solution and potential of the transformed TDS equation are given by formulas (2.10) and (3.2), respectively.

First, assume that the conditions (i) hold. It is immediate that all the matrix functions

$$\exp\{x_r A_r\}, \quad \exp\{\alpha^{-1} t A_r^2\}, \quad \exp\{-x_r B_r\}, \quad \exp\{-\alpha^{-1} t B_r^2\} \quad (1 \leq r \leq k)$$

are matrix polynomials, and thus Φ_0, Ψ_0, and $\mathcal{S}$, which are given by (3.3) and (3.4), are matrix polynomials with respect to x and t. The statement of the proposition follows.

Next, assume that conditions (4.1) hold. Because of (4.1) we have

$$\mathrm{e}_A(x,t) = \mathrm{e}^{f(x,t)} p_A(x,t), \quad \mathrm{e}_B(-x,-t) = \mathrm{e}^{f(x,t)} p_B(x,t), \tag{4.2}$$

where p_A and p_B are matrix polynomials, whereas f and g are scalar polynomials:

$$f(x,t) = \sum_{r=1}^{k} \mu_r x_r + \alpha^{-1} \sum_{r=1}^{k} \mu_r^2 t, \quad g(x,t) = -\sum_{r=1}^{k} \lambda_r x_r - \alpha^{-1} \sum_{r=1}^{k} \lambda_r^2 t. \tag{4.3}$$

Since $\mathcal{S}_0 = 0$, we derive from formulas (3.3), (3.4), and (4.2) that $\Psi_0^* \mathcal{S}^{-1} \Phi_0$ is rational, and so (in view of (3.2)) the potential $\widetilde{q}$ is rational too. ■

Further we assume again that $N_1 = N_2 = N$, that is, R is a square matrix. The following proposition is immediate from (2.10), (3.2)-(3.4).

Proposition 4.2 *Let the conditions of Theorem 3.2 hold and let also equalities*

$$\alpha = \mathrm{i}, \quad \widehat{C}_\Psi = \widehat{C}_\Phi^*, \quad \mathcal{S}_0 = \mathcal{S}_0^*; \tag{4.4}$$

$$C_\Psi = C_\Phi^*, \quad R = R^*; \quad B_r = -A_r^* \quad (1 \leq r \leq k) \tag{4.5}$$

be satisfied. Then we have

$$\mathrm{e}_B(-x,-t) = \mathrm{e}_A(x,t)^*, \quad \Phi_0(x,t) = \Psi_0(x,t), \quad \mathcal{S}(x,t) = \mathcal{S}(x,t)^*. \tag{4.6}$$

Furthermore, if the additional relations

$$R \geq 0, \quad \mathcal{S}_0 > 0 \quad \text{or} \quad R > 0, \quad \mathrm{Rank}(\widehat{C}_\Phi) = n \leq N, \quad \mathcal{S}_0 \geq 0 \tag{4.7}$$

hold, then the inequality $\mathcal{S}(x,t) > 0$ holds too, and so $\mathcal{S}$ is invertible and the solution $\widetilde{\Psi}_0^*$ and potential $\widetilde{q}$ of the transformed TDS are non-singular.

Finally, we consider several concrete examples of non-singular, rational, and lump potentials, where (4.4) and (4.5) hold, and

$$\nu_r = \{\delta_{r-i}\delta_{r-j}\}_{i,j=1}^p, \qquad 1 \leq r \leq k. \tag{4.8}$$

Because of (4.8) and the first and third equalities in (4.5), identity (3.1) for $r = 1$ has the form

$$A_1 R + R A_1^* = \mathcal{Q}, \quad \mathcal{Q} = \mathcal{Q}^*, \quad \text{i.e.,}$$
$$(zI_N - \mathrm{i}A_1)^{-1} R = \mathrm{i}(zI_N - \mathrm{i}A_1)^{-1} \mathcal{Q} (zI_N + \mathrm{i}A_1^*)^{-1} + R(zI_N + \mathrm{i}A_1^*)^{-1}. \tag{4.9}$$

If $\sigma(\mathrm{i}A) \subset \mathbb{C}_+$, we take residues and derive from (4.9) a well-known representation

$$R = \frac{1}{2\pi} \int_{-\infty}^{\infty} (zI_N - \mathrm{i}A_1)^{-1} \mathcal{Q} (zI_N + \mathrm{i}A_1^*)^{-1} dz, \quad R = R^*, \tag{4.10}$$

that is, the second equality in (4.5) follows now from the first and third equalities.

Remark 4.3 *For the case that $\mathcal{Q} \geq 0$, representation* (4.10) *implies $R \geq 0$.*

Example 4.4 *Let $k = p = 3$, $N = 2$, and let A_1 be a 2×2 Jordan cell:*

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \mu_0 I_2 + A_0, \quad \varkappa := \mu_0 + \overline{\mu_0} \neq 0; \tag{4.11}$$

$$A_r = (A_1 - c_r I_2)^{-1}, \quad c_r = -\overline{c_r} \qquad (r = 2, 3). \tag{4.12}$$

It is immediate from (4.11) *and* (4.12) *that*

$$A_r = (\mu_0 - c_r)^{-1} I_2 - (\mu_0 - c_r)^{-2} A_0, \tag{4.13}$$

and (*similar to* (3.12) *but with different choices of A_r and h*) *we put*

$$C_\Psi = C_\Phi^*, \quad C_\Phi = \begin{bmatrix} h & A_2 h & A_3 h \end{bmatrix} = \begin{bmatrix} 0 & -(\mu_0 - c_2)^{-2} & -(\mu_0 - c_3)^{-2} \\ 1 & (\mu_0 - c_2)^{-1} & (\mu_0 - c_3)^{-1} \end{bmatrix}, \tag{4.14}$$

where $h = \mathrm{col}\begin{bmatrix} 0 & 1 \end{bmatrix}$. We recover R from the identity

$$A_1 R + R A_1^* = h h^*, \tag{4.15}$$

which is equivalent (*in view of $B_1 = -A_1^*$,* (4.8), *and* (4.14)) *to relation* (3.1) *for $r = 1$. That is, we rewrite* (4.15) *in the form*

$$\varkappa R + \begin{bmatrix} R_{21} + R_{12} & R_{22} \\ R_{22} & 0 \end{bmatrix} = h h^*, \tag{4.16}$$

where R_{ij} are the entries of R. Using (4.16) (*and starting from recovery of R_{22}*), *we easily get a unique R satisfying* (4.15):

$$R = \begin{bmatrix} 2\varkappa^{-3} & -\varkappa^{-2} \\ -\varkappa^{-2} & \varkappa^{-1} \end{bmatrix}, \quad \det R = \varkappa^{-4}. \tag{4.17}$$

Identities (3.1) *for $r > 1$ easily follow from* (4.16), *and so we obtain an S_3-node $\{3, A, B, R, \nu, C_\Phi, C_\Psi\}$. Moreover, Remark 4.3 and the second equality in* (4.17) *yield*

$$R > 0 \quad \text{for} \quad \varkappa = \mu_0 + \overline{\mu_0} > 0. \tag{4.18}$$

Therefore, the conditions of Theorem 3.2 and Proposition 4.2 are fulfilled.

Since $A_0^2 = 0$, *formula* (4.13) *and second relations in* (3.3) *and* (4.11) *imply*

$$\mathrm{e}_A(x,t) = \mathrm{e}^{\Omega_0(x,t)}(I_2 + \Omega_1(x,t)A_0) = \mathrm{e}^{\Omega_0(x,t)} \begin{bmatrix} 1 & \Omega_1(x,t) \\ 0 & 1 \end{bmatrix}, \tag{4.19}$$

$$\Omega_0(x,t) := \mu_0 x_1 + (\mu_0 - c_2)^{-1}x_2 + (\mu_0 - c_3)^{-1}x_3 - \mathrm{i}\big(\mu_0^2 + (\mu_0 - c_2)^{-2} + (\mu_0 - c_3)^{-2}\big)t, \tag{4.20}$$

$$\Omega_1(x,t) := x_1 - (\mu_0 - c_2)^{-2}x_2 - (\mu_0 - c_3)^{-2}x_3 - 2\mathrm{i}\big(\mu_0 - (\mu_0 - c_2)^{-3} - (\mu_0 - c_3)^{-3}\big)t. \tag{4.21}$$

Corollary 4.5 *Let* $k = p = 3$, $N = 2$, *and* $\varkappa = \mu_0 + \overline{\mu_0} > 0$. *Define matrices* C_Ψ, C_Φ, *and* R *via* (4.14) *and* (4.17). *Choose* $\widehat{C}_\Psi = \widehat{C}_\Phi^*$ *and* $\mathcal{S}_0 > 0$. *Then relations* (2.10) *and* (3.2)-(3.4), *where* e_A *is given by* (4.19)-(4.21) *and* $\mathrm{e}_B(-x,-t) = \mathrm{e}_A(x,t)^*$, *explicitly define non-singular solutions and potentials of TDS.*

The cases $N > 2$ can be treated in the same way.

Example 4.6 *Let* $k = p = 3$ *and* $N = 3$. *Set*

$$A_1 = \mu_0 I_3 + A_0, \quad A_r = (A_1 - c_r I_3)^{-1} = -\sum_{i=0}^{2}(c_r - \mu_0)^{-i-1}A_0^i, \tag{4.22}$$

$$c_r = -\overline{c_r} \quad (r = 2,3); \quad \varkappa := \mu_0 + \overline{\mu_0} \neq 0, \tag{4.23}$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad h = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_\Phi = C_\Psi^* = \begin{bmatrix} h & A_2 h & A_3 h \end{bmatrix}. \tag{4.24}$$

In a way, which is quite similar to Example 4.4, we can show that an S_3*-node* $\{3, A, B, R, \nu, C_\Phi, C_\Psi\}$ *appears, if we put*

$$B_r = -A_r^* \quad (1 \leq r \leq 3), \quad R = \begin{bmatrix} 6\varkappa^{-5} & -3\varkappa^{-4} & \varkappa^{-3} \\ -3\varkappa^{-4} & 2\varkappa^{-3} & -\varkappa^{-2} \\ \varkappa^{-3} & -\varkappa^{-2} & \varkappa^{-1} \end{bmatrix}. \tag{4.25}$$

Moreover, we have $\det R = \varkappa^{-9} \neq 0$*, and so* (4.18) *is valid for* R *of the form* (4.25) *too. Finally, we note that since* $A_0^3 = 0$*, formulas* (3.3) *and* (4.22) *imply*

$$
\begin{aligned}
\mathrm{e}_A(x,t) =& \mathrm{e}^{\Omega_0(x,t)}\big(I_3 + \Omega_1(x,t)A_0 + \Omega_2(x,t)A_0^2\big) \\
=& \mathrm{e}^{\Omega_0(x,t)} \begin{bmatrix} 1 & \Omega_1(x,t) & \Omega_2(x,t) \\ 0 & 1 & \Omega_1(x,t) \\ 0 & 0 & 1 \end{bmatrix}, \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
\Omega_2(x,t) :=& \frac{1}{2}\Omega_1(x,t)^2 + (\mu_0 - c_2)^{-3}x_2 + (\mu_0 - c_3)^{-3}x_3 \\
& - \mathrm{i}\big(1 + (\mu_0 - c_2)^{-4} + (\mu_0 - c_3)^{-4}\big)t. \qquad (4.27)
\end{aligned}
$$

Corollary 4.7 *Let* $k = p = 3$*,* $N = 3$*, and* $\varkappa = \mu_0 + \overline{\mu_0} > 0$*. Define matrices* C_Ψ*,* C_Φ*, and* R *via* (4.24) *and* (4.25)*. Choose* $\widehat{C}_\Psi = \widehat{C}_\Phi^*$ *and* $\mathcal{S}_0 > 0$*. Then relations* (2.10) *and* (3.2)-(3.4)*, where* $\mathrm{e}_B(-x,-t) = \mathrm{e}_A(x,t)^*$ *and* e_A *is given by* (4.26) (*using functions* Ω_i*, which are introduced in* (4.20), (4.21), (4.27))*, explicitly define non-singular solutions and potentials of TDS.*

If we take the S_3-node from Example 4.6 and set $\mathcal{S}_0 = 0$, then relations (4.1) hold. Therefore, taking into account Proposition 4.1 we see that the potential $\widetilde{q}$ is rational. Usually, in the study of *lumps* it is required that the corresponding potentials (or solutions) are not only rational but also non-singular. To choose non-singular $\widetilde{q}$, recall that (4.18) is valid for R of the form (4.25). Hence, in view of Proposition 4.2 conditions

$$
\varkappa = \mu_0 + \overline{\mu_0} > 0, \quad \mathrm{Rank}(\widehat{C}_\Phi) = n \leq 3 \qquad (4.28)
$$

imply that $\widetilde{q}$ is non-singular, and our next corollary follows.

Corollary 4.8 *Let* $k = p = 3$*,* $N = 3$*, and* $\alpha = \mathrm{i}$*. Define matrices* C_Ψ*,* C_Φ*, and* R *via* (4.24) *and* (4.25)*. Let numbers* $\varkappa$*,* μ_0, n *and matrix* $\widehat{C}_\Phi = \widehat{C}_\Psi^*$ *satisfy* (4.28)*, and set* $\mathcal{S}_0 = 0$*. Then relations* (2.10) *and* (3.2)-(3.4)*, where* $\mathrm{e}_B(-x,-t) = \mathrm{e}_A(x,t)^*$ *and* e_A *is given by* (4.26) (*using functions* Ω_i*, which are introduced in* (4.20), (4.21), (4.27))*, explicitly define non-singular solutions and potentials of TDS. Moreover, the potentials* $\widetilde{q}$ *are not only non-singular but also rational.*

5 Conclusion

Thus, a version of the binary Darboux transformation is constructed for TDS equation, where $k \geq 1$. (No binary Darboux transformations for the TDS equation, where $k > 1$, were known before.) This is an iterated GBDT version. New families of non-singular and rational potentials and solutions are obtained. Some results are new for the case that $k = 1$ too.

A certain generalization of a colligation introduced by M.S. Livšic and of the S-node introduced by L.A. Sakhnovich, which we call S-multinode, is used in our construction, and could be useful also in constructions of explicit solutions for other multidimensional systems. Another interesting possibility is application to generalized multidimensional nonlinear Schrödinger equations from [33].

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